DIFFERENTIAL EQUATIONS (DE)

BMA 31

Unit I Ordinary linear Differential Equation

Bernoulli equation – Exact DE – Equations reducible to exact DE – Equations of I order and Higher Degree: Equations solvable for p, for y, and for x – Clairaut's equation.

UNIT II Ordinary Linear D.E

Method of variation of parameters – II order D.E with Constant Coefficients for finding the Particular Integrals of the form $e^{ax}.V$ where V is $\sin mx$ (or) $\cos mx$ and x^n – Equations reducible to linear equations with Constant Coefficients: Cauchy's homogeneous linear equation – Legendre's linear equation.

Unit III Differential Equation of other types

Simultaneous linear D.E with Constant Coefficients – Total D.E – Simultaneous T.D.E. – Equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ – Method of grouping – Method of multipliers.

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II B.Sc MATHEMATICS - III SEMESTER

DIFFERENTIAL EQUATIONS (D.E)

UNIT - I Ordinary Linear Differential Equations.

Bernoulli Equation — Exact Differential equations — Equations Reducible to Exact equation - Equations of first order and higher degree : Equations solvable for p, Equations solvable for x and Equations solvable for y — clairaut's equation.

Definitions

(1) A differential equation is an equation which involves differential coefficients or differentials.

Example: (1) $e^x dx + e^y dy = 0$.

(2) $\dfrac{d^2x}{dt^2} + n^2 x = 0$

(3) $\left.\begin{array}{l}\dfrac{dx}{dt} - wy = a\cos pt, \\ \dfrac{dy}{dt} + wx = a\sin pt\end{array}\right\}$

(4) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$

(5) $y = x\dfrac{dy}{dx} + \dfrac{x}{dy/dx}$

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = c. \rightarrow (6)$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}. \rightarrow (7)$$

(2) An ordinary differential equation is that in which all the differential coefficients have reference to a single independent variable.

Example: equations (1), (2), (3). (5), (6).

(3) A partial differential equation is that in which there are two or more independent variables and partial differential coefficients with respects to any of them.

Example: (4) & (7).

Note: ~~The solution is also~~

(4) The <u>order</u> of a differential equation is the order of the highest derivative appearing in it.

(5) The <u>degree</u> of a differential equation is the degree of the highest derivative occuring in it after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

<u>Examples.</u>

Equation (1) → I order, I degree

" (2) → II order, I "

" (5) → I ", II "

" (6) → II ", II "

(5) ⇒ $y = x\frac{dy}{dx} + \frac{x}{dy/dx}$.

$\Rightarrow y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + x$.

(6) ⇒ $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = c$.

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = c\frac{d^2y}{dx^2}$.

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3} = c\left(\frac{d^2y}{dx^2}\right)^2$.

(6) <u>Solution:</u> A solution (or) integral of a D.E. is a relation between the variables which satisfies the given D.E.

<u>Example</u>. $x = A\cos(nt+\alpha) \rightarrow$ (a)

$\frac{d^2x}{dt^2} + n^2x = 0 \longrightarrow$ (b)

(a) is a solution of (b).

Types of solutions of a D.E.

- General (or) Complete solution
- Particular solution.
- Singular solution.

Note: The solution is called the primitive of the D.E.

General solution:

The general (or) complete solution of a D.E. is that in which the number of arbitrary constants is equal to the order of the D.E.

Example. $x = A\cos(nt+\alpha) \longrightarrow (a)$

$$\frac{d^2x}{dt^2} + n^2x = 0 \longrightarrow (b)$$

(a) is general solution of (b), since number of arbitrary constants in (a) is (A, α) which is same as the order of D.E. (b).

Both are two.

Particular solution:

A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

Example: $x = A\cos(nt+\pi/4)$ is the particular solution of (b) since it can be got by putting $\alpha = \pi/4$.

Singular solution:

A D.E. may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a singular solution.

Equations of the I order & I degree.

Special methods of solution applied to the various types of equations:

Type (1) Variable Separable method

(2) Homogeneous equations.

(3) Linear equations.

(4) Exact equations.

(5) Bernoulli equations. ....

Type (1) Variable ~~separable~~ Separable method.

General form: $f(y)\,dy = \phi(x)\,dx$.

Integrating both sides, $\int f(y)dy = \int \phi(x)\,dx + c$, we get the solution.

Problem (1) Solve $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

Solution: Given equation is $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

$$\Rightarrow \frac{dy}{dx} = e^{3x} \cdot e^{-2y} + x^2 e^{-2y}.$$

$$= e^{-2y}\left(e^{3x} + x^2\right)$$

$$\Rightarrow e^{2y}\,dy = \left(e^{3x} + x^2\right)dx. \quad \text{[variable separated]}$$

Integrating both sides.

$$\int e^{2y}\,dy = \int \left(e^{3x} + x^2\right)dx.$$

$$\Rightarrow \frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c.$$

$$\Rightarrow \boxed{3e^{2y} = 2\left(e^{3x} + x^3\right) + c_1} \quad \text{where } c_1 = 6c.$$

Problem ②

Solve $\frac{dy}{dx} + \left(\frac{1-y^2}{1+x^2}\right)^{1/2} = 0.$

Solution:

$$\frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0.$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0. \text{ (variables separated)}$$

Integrating, $\boxed{\sin^{-1}y + \sin^{-1}x = c}$ we get the solution.

Problem ③ Solve $ydx - xdy + 3x^2y^2e^{x^3}dx = 0.$

Solution:

$$ydx - xdy = -3x^2y^2e^{x^3}dx.$$

$$\Rightarrow \frac{ydx - xdy}{y^2} = -3x^2e^{x^3}dx$$

$$\Rightarrow \left[\frac{ydx - xdy}{y^2}\right] + 3x^2e^{x^3}dx = 0.$$

$$\Rightarrow d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0.$$

Since $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

& $d\left(e^{x^3}\right) = e^{x^3}\cdot(3x^2) = 3x^2e^{x^3}.$

Integrating we get

$$\boxed{\frac{x}{y} + e^{x^3} = c}$$

which is the solution.

H.W.

④ $\sqrt{1+x^2}\,dx + \sqrt{1+y^2}\,dy = 0. \rightarrow$ Solve.

⑤ $e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0 \longrightarrow$ Solve.

Type ② Homogeneous equations.

An expression of the form $a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n \rightarrow ①$ in which every term is of the n^{th} degree, is called a homogeneous function of degree n.

① can be written as

$$x^n \left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\,(y/x)^2 + \dots + a_n (y/x)^n \right].$$

Thus any function $f(x, y)$ which can be expressed in the form $x^n \phi(y/x)$, is called a homogeneous function of degree n in x and y.

Example: $x^3 \cos(y/x)$ is a homogeneous function of degree 3, in x and y.

Form: $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ where $f(x,y)$ and $\phi(x,y)$ are homogeneous functions of the same degree in x and y.

To solve a homogeneous equation,

(i) Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

(ii) Separate the variables v and x.

(iii) Integrate and find the solution.

Problem: ① Solve $y^2 + x^2 \frac{dy}{dx} = xy\frac{dy}{dx} \longrightarrow ⓐ$

Solution: Put $y = vx$; $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

$$\therefore ⓐ \Rightarrow v^2x^2 + x^2\frac{dy}{dx} = x.(vx)\frac{dy}{dx}$$

$$\Rightarrow v^2x^2 + x^2\frac{dy}{dx} = x^2 v\frac{dy}{dx}.$$

Cancelling x^2 on both sides we get

$$v^2 + \frac{dy}{dx} = v\frac{dy}{dx}.$$

$$v^2 + \frac{dy}{dx} - v\frac{dy}{dx} = 0.$$

$$v^2 + \frac{dy}{dx}(1-v) = 0$$

$$\Rightarrow v^2 + (1-v)\left(v + x\frac{dv}{dx}\right) = 0.$$

$$\Rightarrow v^2 + (1-v)v + (1-v)x\frac{dv}{dx} = 0.$$

$$\Rightarrow \cancel{v^2} + v - \cancel{v^2} + (1-v)x\frac{dv}{dx} = 0.$$

$$\Rightarrow v + (1-v)x\frac{dv}{dx} = 0.$$

$$\Rightarrow (1-v)x\frac{dv}{dx} = -v.$$

$$\Rightarrow \frac{(1-v)}{v}dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{(1-v)}{v}dv + \frac{dx}{x} = 0 \text{ (variable separated)}$$

$y = vx$

$v = y/x$

~~Integrating~~ $\Rightarrow$ ~~log~~

$$\frac{1}{v}dv - dv + \frac{dx}{x} = 0$$

Integrating $\Rightarrow \log v - v + \log x = \log c.$

Substituting $y = vx$, we get

$$\log\left(\frac{y}{x}\right) - y/x + \log x = \log c.$$

$$\log\left(\frac{xy}{x}\right) - y/x = \log c.$$

$$\log y - y/x = \log c.$$

$$\log y - \log c = y/x$$

$$\log(y/c) = y/x.$$

$$\Rightarrow y/c = e^{y/x}.$$

$$\Rightarrow \boxed{y = ce^{y/x}}$$

Problem 2 $\quad x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx.$

Solution:

Given equation is $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx. \longrightarrow (1)$

put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}.$

$$\Rightarrow dy = v\,dx + x\,dv.$$

Now (1) $\Rightarrow x\left[v\,dx + x\,dv\right] - vx\,dx = \sqrt{x^2+v^2x^2}\;.dx.$

$$\Rightarrow \cancel{vx\,dx} + x^2dv - \cancel{vx\,dx} = \sqrt{x^2(1+v^2)}\,dx.$$

$$\Rightarrow x^2dv = x\sqrt{1+v^2}\,dx.$$

$$\div x \Rightarrow x\,dv = \sqrt{1+v^2}.dx.$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}\left[\text{variables separated}\right].$$

Integrating, $\log\left[v+\sqrt{1+v^2}\right] = \log x + \log c.$

$$\Rightarrow v + \sqrt{1+v^2} = cx.$$

$$\Rightarrow y/x + \sqrt{1+\frac{y^2}{x^2}} = cx.$$

$$\Rightarrow y/x + \frac{\sqrt{x^2+y^2}}{x} = cx$$

$$\Rightarrow \boxed{y + \sqrt{x^2+y^2} = cx^2}$$

Formula: $\displaystyle\int \frac{dx}{\sqrt{1+x^2}} = \log\left[x + \sqrt{1+x^2}\right].$

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Problem 3:

Solve $(x^2-y^2)\,dx - xy\,dy = 0$.

Solution:

Given equation is $(x^2-y^2)\,dx - xy\,dy = 0$ which is homogeneous equation in x and y. → (1)

Put $y = vx$. $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow dy = v\,dx + x\,dv$.

$\therefore (1) \Rightarrow (x^2 - v^2x^2)\,dx - x(vx)\cdot[v\,dx + x\,dv] = 0$

$\Rightarrow x^2(1-v^2)\,dx - x^2v[v\,dx + x\,dv] = 0$

$\Rightarrow (1-v^2)\,dx = v^2\,dx + xv\,dv.$

$\Rightarrow 1-v^2 = v^2 + xv\frac{dv}{dx}$.

$\Rightarrow 1-v^2-v^2 = xv\frac{dv}{dx}$

$\Rightarrow 1-2v^2 = xv\frac{dv}{dx}$.

$\Rightarrow \frac{dx}{x} = \frac{v}{1-2v^2}\,dv$.

Integrating on both sides, we get

$\int \frac{v}{1-2v^2}\,dv = \int \frac{dx}{x} + c.$

Put $1-2v^2 = t$.
$-4v\,dv = dt$
$v\,dv = -\frac{dt}{4}$.

$\Rightarrow \int -\frac{dt}{4t} = \log x + c.$

$\Rightarrow -\frac{1}{4}\log t = \log x + c.$

$-\frac{1}{4}\log(1-2v^2) = \log x + c.$

$\Rightarrow -\frac{1}{4}\log(1-2v^2) + \log x = -c.$

$\Rightarrow +\frac{1}{4}\log[1-2v^2] + \log x = -c.$

$\Rightarrow 4\log x + \log(1-2v^2) = -4c.$

$v = y/x \Rightarrow 4\log x + \log\left[1-\frac{2y^2}{x^2}\right] = -4c.$

$4\log x + \log\left[1-\frac{2y^2}{x^2}\right] = c_1$ where $c_1 = -4c$.

$\Rightarrow \log x^4 + \log\left(\frac{x^2-2y^2}{x^2}\right)$

$$\Rightarrow \log\left[x^4\left(\frac{x^2-2y^2}{x^2}\right)\right] = c_1.$$

$$\Rightarrow \quad x^2(x^2-2y^2) = e^{c_1} = c_2.$$

$$\Rightarrow \boxed{x^2(x^2-2y^2) = c_2.}$$ is the solution.

Note: (1) $\Rightarrow \quad (x^2-y^2)\,dx = xy\,dy$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2-y^2}{xy} = \frac{f_1(x,y)}{f_2(x,y)}.$$

Problem 4: Solve $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0.$

Solution: Given equation is

$$\left(1+e^{x/y}\right)dx = -e^{x/y}\left(1-x/y\right)dy$$

$$\Rightarrow \cancel{\frac{dy}{dx} = -\frac{1+e^{x/y}}{e^{x/y}}}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-e^{x/y}\left(1-x/y\right)}{1+e^{x/y}} \longrightarrow (1)$$

which is a homogeneous equation.

Put $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}.$

$$\therefore (1) \Rightarrow \quad v + y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v}.$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v.$$

$$= -\left[\frac{e^v - \cancel{ve^v} + v + \cancel{ve^v}}{1+e^v}\right].$$

$$y\frac{dv}{dy} = \cancel{v} - \frac{(v+e^v)}{1+e^v}.$$

Separative the variables

$$\frac{1+e^v}{v+e^v}\,dv = -\frac{dy}{y}.$$

$$\Rightarrow -\frac{dy}{y} = \frac{1+e^v}{v+e^v}\,dv.$$

$$= \frac{d(v+e^v)}{v+e^v}.$$

Integrating, we get

$$-\log y = \log(v+e^v) + c.$$

$$\Rightarrow -c = \log(v+e^v) + \log y.$$

$$\log y(v+e^v) = -c$$

$$\Rightarrow y(v+e^v) = e^{-c}.$$

$$\Rightarrow y\left[\cancel{x/y} + e^{x/y}\right] = e^{-c}.$$

$v = \frac{x}{y}$

$$\Rightarrow \boxed{x + ye^{x/y} = c_1}$$

which is the solution.

Hw

5) Solve: $(x^2-y^2)\,dx = 2xy\,dy$.

6) $\frac{dy}{dx} = \frac{x-y}{x+y}$

7) $(x^2+y^2)\frac{dy}{dx} = xy.$

A D.E. is said to be linear if the dependent variable and its derivatives occur only in the first degree.

The linear equation in standard form of first order, known as Leibnitz's linear equation is

$\boxed{\frac{dy}{dx} + Py = Q}$ where P and Q are functions of x only.

$\hookrightarrow$ ⓐ

ⓐ $\Rightarrow$ Consider $\frac{dy}{dx} + Py = 0.$

$\Rightarrow \frac{dy}{y} + Pdx = 0.$

Integrating $\Rightarrow \log y + \int Pdx = 0$

$\Rightarrow \cancel{\log y = -\int Pdx.}$

$\Rightarrow \cancel{y = e^{-\int Pdx}}$

Solution is $\boxed{ye^{\int Pdx} = c.}$

Solution is $\boxed{ye^{\int Pdx} = \int Q \cdot e^{\int Pdx} dx + c.}$

The factor $e^{\int Pdx}$ on multiplying by L.H.S. of ⓐ becomes the differential coefficient of a simple function, is called the integrating factor (I.F.) of the linear equation ⓐ.

$\therefore \boxed{I.F = e^{\int Pdx}}$

& $\therefore$ Solution is $y(I.F.) = \int Q.(I.F.)\, dx + c$

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Problem (1)

Solve $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$.

Solution:

Given equation is

$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2 \rightarrow (1)$

$\div$ by $(x+1)$, $\Rightarrow$

$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1) \rightarrow (2)$ which is Leibnitz linear equation. $\frac{dy}{dx} + Py = Q$.

Here $P = -\frac{1}{x+1}$, $Q = e^{3x}(x+1)$

Now $\int P\,dx = -\int \frac{1}{x+1}\,dx = -\log(x+1)$

$= \log(x+1)^{-1}$.

$\Rightarrow$ I.F. $= e^{\int P\,dx} = e^{\log(x+1)^{-1}} = (x+1)^{-1}$

$\Rightarrow \boxed{I.F. = \frac{1}{x+1}}$

Solution is $y(IF) = \int Q.(IF)\,dx + c$.

$\Rightarrow \frac{y}{x+1} = \int e^{3x}\frac{\cancel{(x+1)}}{\cancel{(x+1)}}\frac{1}{x+1}\,dx + c$

$= \int e^{3x}\,dx + c$.

$\Rightarrow \frac{y}{x+1} = \frac{e^{3x}}{3} + c$.

$\Rightarrow \boxed{y = \left(\frac{e^{3x}}{3} + c\right)(x+1).}$

Problem (2)

Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy.$

Solution:

Given equation is $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$ $\rightarrow (1)$

(1) has y^2 and $\tan^{-1}y$.

$\Rightarrow$ it is not linear in y.

So (1) can be rewritten as

$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x.$$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}.$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \rightarrow (2)$$ which is linear in x. $\left[\text{Form } \frac{dx}{dy} + Px = Q\right]$.

Here $P = \frac{1}{1+y^2}$

$Q = \frac{\tan^{-1}y}{1+y^2}$.

$$\therefore \text{IF} = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}\,dy}$$

$$\boxed{\text{I.F.} = e^{\tan^{-1}y}}$$

Solution is $x(\text{IF}) = \int Q(\text{IF})\,dy + c.$

put $t = \tan^{-1}y$

$\Rightarrow dt = \frac{dy}{1+y^2}$.

$$\Rightarrow x e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\,dy + c.$$

$$= \int t e^t\,dt + c$$

$$= t(e^t) - \int 1.e^t\,dt + c.$$

$$= te^t - e^t + c.$$

$$= (\tan^{-1}y - 1)e^{\tan^{-1}y} + c.$$

Integration by parts

$\int u\,dv = uv - \int v\,du.$

$u = t,\ dv = e^t dt.$

$$\Rightarrow \boxed{x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}}$$

Problem (3)

Solve $x\frac{dy}{dx} + y\log x = e^x . x^{1-\frac{1}{2}\log x}$.

Solution: Given equation is

$$x\frac{dy}{dx} + y\log x = e^x . x^{1-\frac{1}{2}\log x} \longrightarrow (1)$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x}\log x = e^x . x^{-\frac{1}{2}\log x} \longrightarrow (2)$$

$x^{1-\frac{1}{2}\log x} = x^1 . x^{-\frac{1}{2}\log x}$.

which is the standard form

$\frac{dy}{dx} + Py = Q$.

Here $P = \frac{\log x}{x}$, $Q = e^x . x^{-\frac{1}{2}\log x}$.

$I.F. = e^{\int P dx}$

$\int P dx = \int \frac{\log x}{x} dx$.

put $u = \log x$
$\Rightarrow du = \frac{1}{x} dx$.

$= \int u\, du = \frac{u^2}{2}$

$= \frac{(\log x)^2}{2}$.

$\therefore I.F = e^{\int P dx} = e^{\frac{(\log x)^2}{2}} = \left(e^{\log x}\right)^{\frac{\log x}{2}} = x^{\frac{\log x}{2}}$

$\therefore \boxed{IF = x^{\frac{\log x}{2}}}$

Soln. is $y(IF) = \int Q(IF)dx + c$.

$\Rightarrow y . x^{\frac{\log x}{2}} = \int e^x . x^{-\frac{1}{2}\log x} . x^{\frac{\log x}{2}} dx + c$.

$= \int e^x dx + c = e^x + c$.

$\therefore$ solution is $\boxed{y . x^{\frac{\log x}{2}} = e^x + c}$

Problem (4) (5)

Solve $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$.

Solution: form $\frac{dy}{dx} + Py = Q$.

$\Rightarrow P = \cos x, \quad Q = \frac{\sin 2x}{2}$.

$IF = e^{\int P dx} = e^{\int \cos x\, dx} = e^{\sin x}$

$\Rightarrow \boxed{IF = e^{\sin x}}$

Solution is $y(IF) = \int Q(IF)\,dx + c$.

$\Rightarrow ye^{\sin x} = \int \frac{\sin 2x}{2} \cdot e^{\sin x} dx + c$

$=$

$z = \sin x$.

$dz = \cos x\, dx$.

HW. (5)

BERNOULLI'S EQUATION [Jakob Bernoulli -

The equation $\frac{dy}{dx} + Py = Qy^n \longrightarrow (1)$ where P and Q are functions of x only, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation.

To solve (1), divide both sides by y^n;

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + \frac{P.y}{y^n} = \frac{Qy^n}{y^n}.$$

$$\Rightarrow y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \longrightarrow (2).$$

Put $y^{1-n} = z. \Rightarrow (1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \boxed{y^{-n}\frac{dy}{dx} = \frac{1}{n-1}\frac{dz}{dx}}$

$$\therefore (2) \Rightarrow \frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

$$\Rightarrow \frac{dz}{dx} + P(1-n)z = Q(1-n) \longrightarrow (3)$$

which is Leibnitz linear equation in z.

(3) can be solved to get the required answer.

~~Prob~~

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Problem ①

Solve $\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$.

Solution: Given equation is $\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2} \rightarrow ①$

Multiplying by y^2,

$① \Rightarrow y^2 \frac{dy}{dx} - y^3 \tan x = \sin x \cos^2 x \rightarrow ②$

① form is $\frac{dy}{dx} + Py = Qy^n$.

Put $z = y^3$. $\Rightarrow \frac{dz}{dx} = 3y^2 \frac{dy}{dx}$.

$\Rightarrow y^2 \frac{dy}{dx} = \frac{1}{3} \frac{dz}{dx}$.

$\therefore ② \Rightarrow \frac{1}{3}\frac{dz}{dx} - z.\tan x = \sin x \cos^2 x$

$\Rightarrow \frac{dz}{dx} - 3z\tan x = 3\sin x \cdot \cos^2 x \rightarrow ③$

which is Leibnitz Linear equation in z of the form $\frac{dz}{dx} + Pz = Q$.

Here $P = -3\tan x$, $Q = 3\sin x \cos^2 x$.

$I.F = e^{\int P dx} = e^{\int -3\tan x \, dx} = e^{\int -3 \frac{\sin x}{\cos x} dx}$

$= e^{3\log \cos x} = e^{\log \cos^3 x} = \cos^3 x$

$\boxed{IF = \cos^3 x}$

Solution is $z\,(IF) = \int Q\,(IF)\,dx + c$.

$\Rightarrow z\cos^3 x = \int 3\sin x \cos^2 x \cdot \cos^3 x \, dx + c$.

$= 3\int \sin x \cos^5 x \, dx + c$.

$= -\frac{3\cos^6 x}{6} + c$

$= -\frac{\cos^6 x}{2} + c$

$$\boxed{y^3 \cos^3 x = -\frac{\cos^6 x}{2} + c}$$

Problem 2

Solve : $(x+1)\frac{dy}{dx}+1 = 2e^{-y}$.

Solution: Given equation is $(x+1)\frac{dy}{dx}+1=2e^{-y} \rightarrow$ ①

Multiplying by e^{y},

$e^{y}(x+1)\frac{dy}{dx} + e^{y} = 2. \rightarrow$ ② [Bernoulli's form]

Put $z = e^{y} \Rightarrow \frac{dz}{dx} = e^{y}\frac{dy}{dx}$.

$\therefore$ ② $\Rightarrow (x+1)\frac{dz}{dx} + z = 2.$

$\Rightarrow \frac{dz}{dx} + \frac{1}{x+1}\cdot z = \frac{2}{x+1} \rightarrow$ ③

which is Leibnitz linear equation in z.

Here $P = \frac{1}{x+1}$, $Q = \frac{2}{x+1}$.

$\int P dx = \int \frac{1}{x+1} dx = \log(x+1)$.

$\therefore$ I.F $= e^{\int P dx} = e^{\log(x+1)} = x+1$

$\therefore \boxed{IF = x+1}$

Solution is $z(IF) = \int Q.(IF)\,dx + c$.

$\Rightarrow z(x+1) = \int \frac{2}{x+1}(x+1)\,dx + c$.

$= 2x + c$.

$\Rightarrow \boxed{e^{y}(x+1) = 2x+c}$

Problem 3

Solve $x\frac{dy}{dx} + y = x^3y^6$.

Solution: Given equation is

$$x\frac{dy}{dx} + y = x^3y^6 \longrightarrow (1)$$

Divide by xy^6, $\Rightarrow y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \longrightarrow (2)$

Put $y^{-5} = z \Rightarrow -5y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$.

$$\therefore (2) \Rightarrow -\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2.$$

$$\Rightarrow \frac{dz}{dx} - \frac{5}{x}z = -5x^2 \longrightarrow (3)$$

which is Leibnitz linear equation in z.

$P = -\frac{5}{x}$, $Q = -5x^2$.

$$\int P\,dx = \int -\frac{5}{x}\,dx = -5\log x. = \log x^{-5}$$

$$\therefore I.F. = e^{\int P\,dx} = e^{\log x^{-5}} = x^{-5}.$$

$$\boxed{I.F. = \frac{1}{x^5}}$$

Solution is ~~y(~~ $z(I.F) = \int Q(I.F)\,dx + c$.

$$\Rightarrow z\cdot\frac{1}{x^5} = \int -5x^2\cdot\frac{1}{x^5}\,dx + c.$$

$$= \int -5x^{-3}\,dx + c.$$

$$= -5\cdot\frac{x^{-2}}{-2} + c.$$

$$\Rightarrow y^{-5}x^{-5} = +\frac{5}{2}x^{-2} + c.$$

Dividing by $y^{-5}x^{-5} \Rightarrow 1 = \frac{5}{2}\cdot x^3y^5 + cx^5y^5$.

$$\Rightarrow \boxed{(5 + 2cx^2)x^3y^5 = 2}$$

which is the required solution.

Problem (4)

Solve $xy(1+xy^2)\frac{dy}{dx} = 1$.

Solution: Given equation is $xy(1+xy^2)\frac{dy}{dx} = 1 \rightarrow (1)$

$\Rightarrow (xy+x^2y^3)\frac{dy}{dx} = 1 \Rightarrow xy+x^2y^3 = \frac{dx}{dy}$.

$\Rightarrow \frac{dx}{dy} - xy = x^2y^3 \rightarrow (2)$.

Dividing by x^2, $x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \rightarrow (3)$.

Put $z = x^{-1} \Rightarrow \frac{dz}{dy} = -x^{-2}\frac{dx}{dy}$.

$\Rightarrow -\frac{dz}{dy} - yz = y^3$.

$\Rightarrow \frac{dz}{dy} + yz = -y^3 \rightarrow (4)$ which is Leibnitz linear in z.

$P = y \Rightarrow \int P\,dy = \int y\,dy = y^2/2$.

$\therefore$ I.F. $= e^{\int P dy} = e^{y^2/2}$.

Solution is $z(IF) = \int Q(IF)\,dy + c$

$\Rightarrow ze^{y^2/2} = \int -y^3 \cdot e^{y^2/2}\,dy + c$.

put $\frac{y^2}{2} = t$.

$\Rightarrow y\,dy = dt$.

$y^3 e^{\frac{y^2}{2}} dy = y^2 \cdot e^{y^2/2}\,y\,dy$.

$= 2t \cdot e^t \cdot dt$.

$= -\int 2t \cdot e^t\,dt + c$.

$= -2\int te^t\,dt + c$

$= -2\left[t(e^t) - 1 \cdot e^t\right] + c$.

$= -2(te^t - e^t) + c = -2(t-1)e^t + c$.

$\Rightarrow ze^{y^2/2} = -2\left(\frac{y^2}{2} - 1\right)e^{y^2/2} + c = (2-y^2)e^{y^2/2} + c$.

$\Rightarrow z = (2-y^2) + ce^{-y^2/2}$.

$\Rightarrow \boxed{\frac{1}{x} = (2-y^2) + ce^{-y^2/2}}$

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